On Blow-up of a Seimilinear Heat Equation with Nonlinear Boundary Conditions

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Abstract

This paper deals with the blow-up properties of the solutions of the semilinear heat equation $u_t = \Delta u + \lambda e^{pu}$ in $B_R \times (0,T)$ with the nonlinear boundary conditions $\frac{\partial u}{\partial \eta} = e^{qu}$ on $\partial B_R \times (0,T)$, where B_R is a ball in R^n, η is the outward normal, $p > 0, q > 0,\ \lambda > 0$. The upper and lower blow-up rate estimates are established. It is also proved under some restricted assumptions, that the blow-up occurs only on the boundary.

1 Introduction

In this paper, we consider the initial-boundary value problem

$$\left.\begin{array}{ll} u_t = \Delta u + \lambda e^{pu}, & (x,t) \in B_R \times (0,T), \\ \frac{\partial u}{\partial \eta} = e^{qu}, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & x \in B_R, \end{array}\right\} \tag{1.1}$$

where $p > 0, q > 0,\ \lambda > 0$, B_R is a ball in R^n, η is the outward normal, u_0 is nonnegative, radially symmetric, nondecreasing, smooth function satisfies the conditions

$$\frac{\partial u_0}{\partial \eta} = e^{qu_0}, \qquad x \in \partial\Omega, \tag{1.2}$$

$$\Delta u_0 + \lambda e^{pu_0} \geq 0, \quad u_{0r}(|x|) \geq 0, \quad x \in \overline{\Omega}_R. \tag{1.3}$$

The problem of the semilinear heat equation with nonlinear boundary conditions:

$$\left.\begin{array}{ll} u_t = \Delta u + \lambda f(u), & (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} = g(u), & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{array}\right\} \tag{1.4}$$

has been studied by many authors (see for example [1, 10, 6]). The crucial point of these works was the question whether the reaction term in the semilinear equation can prevent (affect) blow-up. For instance, in [1] it has been studied the blow-up solutions of problem (1.4), where $\lambda < 0$ and

$$f(u) = u^p, \quad g(u) = u^q, \quad p, q > 1, \tag{1.5}$$

for $n = 1$ or $\Omega = B_R$. Particularly, it was shown that the exponent $p = 2q - 1$ is critical for blow-up in the following sense:

(i) If $p < 2q - 1$ (or $p = 2q - 1$ and $-\lambda < q$), then there exist solutions, which blow up in finite time and the blow-up occurs only on the boundary.

(ii) If $p > 2q - 1$ (or $p = 2q - 1$ and $-\lambda > q$), then all solutions exist globally and are globally bounded.

In [9] J. D. Rossi has proved for the case (i), where $n = 1$, $\Omega = [0, 1]$, that there exist positive constants C, c such that the upper (lower) blow-up rate estimate take the following forms

$$c \leq \max_{[0,1]} u(\cdot, t)(T - t)^{\frac{1}{2(q-1)}} \leq C, \quad 0 < t < T.$$

In [6] it has been studied another special case of problem (1.4), where $\lambda = 1$, f, g as in (1.5), $\Omega = [0, 1]$ or it is a bounded domain with C^2 boundary, it was proved that the solutions of (1.4) exist globally if and only if $\max\{p, q\} \leq 1$, otherwise, every solution has to blow up in finite time. Moreover, the blow-up occurs only on the boundary. The blow-up rate estimate for this case has been studied in [6, 9], for $n = 1, \Omega = [0, 1]$, it has been shown that there exist positive constants c, C such that

$$c \leq \max_{[0,1]} u(\cdot, t)(T - t)^{\alpha} \leq C, \quad 0 < t < T,$$

where $\alpha = 1/(p - 1)$ if $p \geq 2q - 1$, and $\alpha = 1/[2(q - 1)]$ if $p < 2q - 1$.

We observe that if $p < 2q - 1$, then the nonlinear term at the boundary determines and gives the blow-up rate while, if $p > 2q - 1$, then the reaction term in the semilinear equation dominates and gives the blow-up rate.

Later, in [10] it was considered a second special case of (1.4), where $\lambda = -a, a > 0$, f, g are of exponential forms, namely

$$\left.\begin{array}{ll} u_t = \Delta u - a e^{pu}, & (x, t) \in \Omega \times (0, T), \\ \frac{\partial u}{\partial \eta} = e^{qu}, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{array}\right\} \tag{1.6}$$

where $p, q > 0$, u_0 satisfies (1.2), (1.3).

It has been shown that in case of Ω is a bounded domain with smooth boundary, the critical exponent can be given as follows

(i) If $2q < p$, the solutions of problem (1.6) are globally bounded.

(ii) If $2q > p$, the solutions of problem (1.6) blow up in finite time for large initial data.

(iii) If $2q = p$, the solutions may blow up in finite time for large initial data.

Moreover, in case $\Omega = B_R$, the blow-up occurs only on the boundary and there exist positive constants c, C such that the upper (lower) blow-up rate estimate take the following form

$$\log C_1 - \frac{1}{2q}\log(T-t) \leq \max_{\overline{B}} u(\cdot,t) \leq \log C_2 - \frac{1}{2q}\log(T-t), \quad 0 < t < T.$$

Therefore, the blow-up properties (blow-up location and bounds) of problem (1.6) are the same as that of problem (1.6), where $a = 0$, which has been considered in [2].

In this paper, we study the blow-up solutions of problem (1.1). The upper (lower) blow-up rate estimates is obtained. Moreover, under some restricted assumptions, we prove that blow-up occurs only on the boundary.

2 Preliminaries

Since $f(u) = \lambda e^{pu}$, $g(u) = e^{qu}$ are smooth functions, and problem (1.1) is uniformly parabolic, also u_0 satisfies the compatibility condition (1.2), it follows that the existence and uniqueness of local classical solutions to problem (1.1) are known by the standard theory [5]. On the other hand, the nontrivial solutions of this problem blow up in finite time and the blow-up set contains ∂B_R, and that due to comparison principle, [7], and the known blow-up result of problem (1.1), where $\lambda = 0$ (see[2]).

In this paper, we denote for simplicity $u(x,t) = u(r,t)$. The following lemma shows some properties of the classical solutions to problem (1.1).

Lemma 2.1. *Let u be a classical solution to problem (1.1), where u_0 satisfies the assumptions (1.2), (1.3).Then*

(i) $u > 0$, *radial in* $\overline{B}_R \times (0,T)$.

(ii) $u_r \geq 0$, *in* $[0,R] \times [0,T)$.

(iii) $u_t > 0$ *in* $\overline{B}_R \times (0,T)$.

3 Blow-up Rate Estimates

Since $u_r \geq 0$, in $[0,R] \times (0,T)$, it follows that

$$\max_{\overline{B}_R} u(\cdot,t) = u(R,t), \quad 0 < t < T.$$

Therefore, it is sufficient to derive the upper (lower) bounds of blow-up rate for $u(R,t)$.

Theorem 3.1. *Let u be a solution to problem (1.1), where u_0 satisfies the assumptions (1.2), (1.3), T is the blow-up time. Then there is a positive constant c such that*

$$\log c - \frac{1}{2\alpha}\log(T-t) \le u(R,t), \quad t \in (0,T),$$

where $\alpha = \max\{p,q\}$.

Proof. Define

$$M(t) = \max_{\overline{B}_R} u(\cdot,t) = u(R,t), \quad \text{for} \quad t \in [0,T).$$

Clearly, $M(t)$ is increasing in $(0,T)$ (due to $u_t > 0$, for $t \in (0,T),\ x \in \overline{B}_R$). As in [10], for $0 < z < t < T, x \in B_R$, the integral equation of problem (1.1) with respect to u, can be written as follows

$$\begin{aligned} u(x,t) \;=\; & \int_{B_R} \Gamma(x-y,t-z)u(y,z)dy + \lambda \int_z^t \int_{B_R} \Gamma(x-y,t-\tau)e^{pu(y,\tau)}dyd\tau \\ & + \int_z^t \int_{S_R} \Gamma(x-y,t-\tau)e^{qu(y,\tau)}ds_y d\tau \\ & - \int_z^t \int_{S_R} u(y,\tau)\frac{\partial \Gamma}{\partial \eta_y}(x-y,t-\tau)ds_y d\tau, \end{aligned} \tag{3.1}$$

where Γ is the fundamental solution of the heat equation, namely

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{(n/2)}}\exp[-\frac{|x|^2}{4t}]. \tag{3.2}$$

Since $u(y,t) \le u(R,t)$ for $y \in \overline{B}_R$, so, the last equation becomes

$$\begin{aligned} u(x,t) \;\le\; & u(R,z)\int_{B_R} \Gamma(x-y,t-z)dy + \lambda \int_z^t e^{pu(R,\tau)} \int_{B_R} \Gamma(x-y,t-\tau)dyd\tau. \\ & + \int_z^t e^{qu(R,\tau)} \int_{S_R} \Gamma(x-y,t-\tau)ds_y d\tau \\ & + \int_z^t u(R,\tau) \int_{S_R} \mid \frac{\partial \Gamma}{\partial \eta_y}(x-y,t-\tau) \mid ds_y d\tau. \end{aligned}$$

Since u is a continuous function on $\overline{B}_R$, the last inequality leads to

$$\begin{aligned} M(t) \quad \leq \quad & M(z)\int_{B_R}\Gamma(x-y,t-z)dy+\lambda e^{pM(t)}\int_z^t\int_{B_R}\Gamma(x-y,t-\tau)dyd\tau \\ & +e^{qM(t)}\int_z^t\int_{S_R}\Gamma(x-y,t-\tau)ds_yd\tau \\ & +M(t)\int_z^t\int_{S_R}\mid\frac{\partial\Gamma}{\partial\eta_y}(x-y,t-\tau)\mid ds_yd\tau. \end{aligned} \tag{3.3}$$

It is known from [3, 7] that for $0<t_1<t_2,\ x,y\in R^n$, Γ satisfies

$$\int_{B_R}\Gamma(x-y,t_2-t_1)dy\leq 1.$$

Moreover, there exist positive constants k_1,k_2 such that

$$\begin{aligned} &\Gamma(x-y,t_2-t_1)\leq\frac{k_1}{(t_2-t_1)^{\mu_0}}\cdot\frac{1}{|x-y|^{n-2+\mu_0}}, \quad 0<\mu_0<1, \\ &\mid\frac{\partial\Gamma}{\partial\eta_y}(x-y,t_2-t_1)\mid\leq\frac{k_2}{(t_2-t_1)^{\mu}}\cdot\frac{1}{|x-y|^{n+1-2\mu-\sigma}}, \quad \sigma\in(0,1),\ \mu\in(1-\frac{\sigma}{2},1). \end{aligned}$$

If we choose $\mu_0=1/2$, then from [3], there exist positive constants d_1,d_2 such that

$$\int_{S_R}\frac{ds_y}{|x-y|^{n-2+\mu_0}}\leq d_1, \qquad \int_{S_R}\frac{ds_y}{|x-y|^{n+1-2\mu-\sigma}}\leq d_2.$$

From above it follows that there exist $C_1,C_2>0$ such that, the inequality (3.3) becomes

$$M(t)\leq M(z)+\lambda e^{pM(t)}(t-z)+C_1e^{qM(t)}\sqrt{t-z}+C_2M(t)(t-z)^{1-\mu}.$$

Since $t-z\leq T-z$, it follows that

$$M(t)\leq M(z)+\lambda e^{pM(t)}\sqrt{T-z}+C_1e^{qM(t)}\sqrt{T-z}+C_2M(t)(T-z)^{1-\mu}, \tag{3.4}$$

provided $(T-z)\leq 1$.

Clearly,

$$\frac{M(t)}{e^{\alpha M(t)}}\longrightarrow 0, \quad \text{when} \quad t\to T.$$

Thus

$$\frac{M(t)}{e^{\alpha M(t)}}\leq(T-z)^{\frac{1}{2}-(1-\mu)}, \text{ for } t \text{ close to } T.$$

Therefore, the inequality (3.4) becomes

$$M(t)\leq M(z)+\lambda e^{pM(t)}\sqrt{T-z}+C_1e^{qM(t)}\sqrt{T-z}+C_2e^{\alpha M(t)}\sqrt{T-z},$$

thus there is a constant C^* such that

$$M(t) \le M(z) + C^* e^{\alpha M(t)} \sqrt{T-z}, \quad z < t < T, \ t \text{ close to } T.$$

For any z close to T, we can choose $z < t < T$ such that

$$M(t) - M(z) = C_0 > 0,$$

which implies

$$C_0 \le C^* e^{\alpha M(z) + \alpha C_0} \sqrt{T-z}.$$

Thus

$$\frac{C_0}{C^* e^{(\alpha C_0)} \sqrt{T-z}} \le e^{\alpha u(R,z)}.$$

Therefore, there exist a positive constant c such that

$$\log c - \frac{1}{2\alpha} \log(T-t) \le u(R,t), \quad t \in (0,T).$$

□

The next theorem shows similar results to Theorem 3.1 with adding more restricted assumptions on q and u_0. The proof relies on the maximum principle rather than the integral equation.

Theorem 3.2. *Let u be a solution to problem (1.1), where $q \ge 1, T$ is the blow-up time, u_0 satisfies the assumptions (1.2), (1.3), moreover, it satisfies the following condition*

$$u_{0r}(r) - \frac{r}{R} e^{u_0(r)} \ge 0, \quad r \in [0,R]. \tag{3.5}$$

Then there is a positive constant c such that

$$\log c - \frac{1}{2\alpha} \log(T-t) \le u(R,t), \quad t \in (0,T),$$

where $\alpha = \max\{p,q\}$.

Proof. Define the functions J as follows:

$$J(x,t) = u_r(r,t) - \frac{r}{R} e^{u(r,t)}, \quad x \in B_R \times (0,T).$$

A direct calculation shows

$$\begin{aligned}
J_t &= u_{rt} - \frac{r}{R} e^u [u_{rr} + \frac{n-1}{r} u_r + \lambda p e^{pu}], \\
J_r &= u_{rr} - \frac{r}{R} e^u u_r - \frac{1}{R} e^u, \\
J_{rr} &= [u_{rt} - \frac{n-1}{r} u_{rr} + \frac{n-1}{r^2} u_r - \lambda p e^{pu} u_r] \\
&\quad - \frac{r}{R} [e^u u_{rr} + e^u u_r^2] - \frac{2}{R} e^u u_r.
\end{aligned}$$

From above it follows that

$$J_t - J_{rr} - \frac{n-1}{r}J_r = -\frac{n-1}{r^2}[u_r - \frac{r}{R}e^u] + \lambda pe^{pu}[u_r - \frac{r}{R}e^u] + \frac{r}{R}e^u u_r^2 + \frac{2}{R}e^u u_r.$$

Thus

$$J_t - \Delta J - bJ = \frac{r}{R}e^u u_r^2 + \frac{2}{R}e^u u_r \geq 0,$$

for $(x,t) \in B_R \times (0,T) \cap \{r > 0\}$, where $b = [\lambda pe^{pu} - \frac{n-1}{r^2}]$.

Clearly, from (3.5), it follows that

$$J(x,0) \geq 0, \quad x \in B_R,$$

and

$$J(0,t) = u_r(0,t) \geq 0, \; J(R,t) = 0 \quad t \in (0,T).$$

Since

$$\sup_{(0,R)\times(0,t]} b < \infty, \quad \text{for} \quad t < T,$$

from above and maximum principle [8], it follows that

$$J \geq 0, \quad (x,t) \in B_R \times (0,T).$$

Moreover,

$$\frac{\partial J}{\partial \eta}|_{\partial B_R} \leq 0.$$

This means

$$(u_{rr} - \frac{r}{R}e^u u_r - \frac{1}{R}e^u)|_{\partial B_R} \leq 0.$$

Thus

$$u_t \leq (\frac{n-1}{r}u_r + \lambda pe^{pu} + e^u u_r + \frac{1}{R}e^u)|_{\partial B_R}.$$

which implies that

$$u_t(R,t) \leq \frac{n-1}{R}e^{qu(R,t)} + \lambda pe^{pu(R,t)} + e^{(1+q)u(R,t)} + \frac{2}{R}e^{u(R,t)}, \quad t \in (0,T).$$

Thus, there exist a constant C such that

$$u_t(R,t) \leq Ce^{2\alpha u(R,t)}, \quad t \in (0,T).$$

Integrate this inequality from t to T and since u blows up at R, it follows

$$\frac{c}{(T-t)^{\frac{1}{2}}} \leq e^{\alpha u(R,t)}, \quad t \in (0,T)$$

or

$$\log c - \frac{1}{2\alpha}\log(T-t) \leq u(R,t), \quad t \in (0,T).$$

$\square$

Remark 3.3. From Theorems 3.1 and 3.2 we conclude that, when $q > p$ the boundary term plays the dominating role and the lower blow-up rate takes the form:

$$\log c - \frac{1}{2q}\log(T-t) \leq u(R,t), \quad t \in (0,T),$$

moreover, this estimate is coincident with lower blow-up rate estimate of problem (1.1), where $\lambda = 0$, which has been considered in [2], while when $p > q$ the reaction term is dominated and gives the lower blow-up rate as follows

$$\log c - \frac{1}{2p}\log(T-t) \leq u(R,t), \quad t \in (0,T).$$

We next consider the upper bound

Theorem 3.4. *Let u be a solution of problem (1.1), where T is the blow-up time, u_0 satisfies the assumptions (1.2), (1.3) moreover, assume that*

$$\Delta u_0 + f(u_0) \geq a > 0, \quad in\ \overline{B}_R. \tag{3.6}$$

Then there is a positive constant C such that

$$u(R,t) \leq \log C - \frac{1}{q}\log(T-t), \quad t \in (0,T). \tag{3.7}$$

Proof. Define the function J as follows

$$J(x,t) = u_t(r,t) - \varepsilon u_r(r,t), \quad (x,t) \in B_R \times (0,T).$$

Since $u_0(r)$ is bounded in B_R, and by (3.6), for some $\varepsilon > 0$, we have

$$J(x,0) = \Delta u_0(r) + f(u_0(r)) - \varepsilon u_{0r}(r) \geq 0, \quad x \in \overline{B}_R.$$

A simple computation shows

$$\begin{aligned}
J_t &= u_{rrt} + \frac{n-1}{r}u_{rt} + \lambda p e^{pu}u_t - \varepsilon u_{rt}, \\
J_r &= u_{tr} - \varepsilon u_{rr}, \\
J_{rr} &= u_{trr} - \varepsilon u_{tr} + \varepsilon\frac{n-1}{r}u_{rr} - \varepsilon\frac{(n-1)}{r^2}u_r + \varepsilon\lambda p e^{pu}u_r.
\end{aligned}$$

From above, it follows that

$$J_t - J_{rr} - \frac{n-1}{r}J_r - \lambda p e^{pu}J = \varepsilon\frac{(n-1)}{r^2}u_r \geq 0,$$

i.e.

$$J_t - \Delta J - \lambda p e^{pu}J \geq 0, \quad (x,t) \in B_R \times (0,T).$$

Moreover,

$$
\begin{aligned}
\frac{\partial J}{\partial \eta}|_{x\in\partial B_R} &= u_{rt}(R,t) - \varepsilon u_{rr}(R,t) \\
&= q e^{qu(R,t)} u_t - \varepsilon [u_t(R,t) - \frac{n-1}{r} u_r(R,t) - \lambda e^{pu(R,t)}] \\
&\geq [q e^{qu(R,t)} - \varepsilon] u_t(R,t)
\end{aligned}
$$

Since, $u_t > 0$ in $\overline{B}_R \times (0,T)$, if follows that

$$
\frac{\partial J}{\partial \eta} \geq 0, \quad \text{on} \quad \partial B_R \times (0,T),
$$

provided $\varepsilon \leq q e^{\{qu_0(R)\}}$.

Since e^{pu} is bounded on $B_R \times (0,t]$ for $t < T$, from maximum principle [7] and above, we have

$$
J \geq 0, \quad (x,t) \in \overline{B}_R \times (0,T).
$$

In particular, $J(x,t) \geq 0$ for $x \in \partial B_R$, that is

$$
u_t(R,t) \geq \varepsilon u_r(R,t) = \varepsilon e^{qu(R,t)}, \quad t \in (0,T).
$$

Upon integration the above inequality from t to T and since u blows up at R, it follows that

$$
e^{qu(R,t)} \leq \frac{1}{q\varepsilon(T-t)}, \quad t \in (0,T),
$$

or

$$
u(R,t) \leq \log C - \frac{1}{q} \log(T-t), \quad t \in (0,T).
$$

□

Remark 3.5. The upper blow-up rate estimate for problem (1.1), which has been derived in Theorem 3.4, is governed by the boundary term even in case $p > q$. On the other hand, it is known that the upper blow-up bound of problem (1.1), where $\lambda = 0$ (see [2]) takes the form:

$$
u(R,t) \leq \log \frac{C}{(T-t)^{\frac{1}{2q}}}.
$$

Therefore, we conclude that the presence of the reaction term has an important effect on the upper blow-up rate estimate.

4 Blow-up Set

We shall prove in this section that the blow-up to problem (1.1) occurs only on the boundary, restricting ourselves to the special case $p = q = 1$ with some restriction assumption on λ.

Theorem 4.1. *Suppose that the function $u(x,t)$ is $C^{2,1}(\overline{B}_R \times [0,T))$, and satisfies*

$$\left.\begin{array}{ll} u_t = \Delta u + \lambda e^u, & (x,t) \in B_R \times (0,T), \\ u(x,t) \le \log \frac{C}{(T-t)}, & (x,t) \in \overline{B}_R \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{array}\right\}$$

where

$$\lambda[4R^2(n+1)+1] \le \min\left\{\frac{1}{C}, \frac{4(n+1)}{[R^2+4(n+1)T]} e^{-||u_0||_\infty}\right\}, \tag{4.1}$$

$C < \infty$. Then for any $0 \le a < R$, there exist a positive constant A such that

$$u(x,t) \le \log[\frac{1}{A(R^2-r^2)^2}] < \infty \quad \textit{for} \quad 0 \le |x| \le a < R, 0 < t < T.$$

Proof. Let

$$\begin{aligned} v(x) &= A(R^2-r^2)^2, \quad r = |x|, \quad 0 \le r \le R, \\ z(x,t) = z(r,t) &= \log \frac{1}{[v(x)+B(T-t)]}, \quad \text{in} \quad \overline{B}_R \times (0,T), \end{aligned}$$

where $B > 0, A \ge \lambda$.

A direct calculation shows that

$$\begin{aligned} z_t &= \frac{B}{[v(x)+B(T-t)]}, \\ z_r &= \frac{4rA(R^2-r^2)}{[v(x)+B(T-t)]}, \\ z_{rr} &= \frac{[v(x)+B(T-t)][4A(R^2-3r^2)]+16A^2r^2(R^2-r^2)^2}{[v(x)+B(T-t)]^2}. \end{aligned}$$

Thus

$$\begin{aligned} z_t - z_{rr} - \frac{n-1}{r} z_r - \lambda e^z &= \frac{[B-4A(n-1)(R^2-r^2)-\lambda][v(x)+B(T-t)]}{[v(x)+B(T-t)]^2} \\ &\quad - \frac{[4A(R^2-3r^2)][v(x)+B(T-t)]+16Ar^2v(x)}{[v(x)+B(T-t)]^2} \\ &\ge \frac{[B-4A(n-1)(R^2-r^2)-\lambda-4A(R^2-3r^2)-16Ar^2]v(x)}{[v(x)+B(T-t)]^2} \\ &\ge \frac{[B-4AR^2n-4AR^2-\lambda]v(x)}{[v(x)+B(T-t)]^2} \\ &\ge \frac{[B-4AR^2n-4AR^2-A]v(x)}{[v(x)+B(T-t)]^2} \ge 0 \end{aligned}$$

provided

$$B \geq A[4R^2(n+1)+1].$$

i.e.

$$z_t - \Delta z - \lambda e^z \geq 0, \quad \text{in} \quad B_R \times (0,T)$$

Moreover,

$$\begin{array}{llll} z(x,0) = \log \frac{1}{[v(x)+BT]} & \geq \log \frac{1}{[AR^4+BT]} & \geq u(x,0), & x \in B_R, \\ z(R,t) = \log \frac{1}{B(T-t)} & \geq \log \frac{C}{(T-t)} & \geq u(R,t), & t \in (0,T) \end{array}$$

provided

$$B \leq \min \left\{ \frac{1}{C}, \frac{4(n+1)}{R^2+4(n+1)T} e^{-||u_0||_\infty} \right\}.$$

From above, and the comparison principle [7], we obtain

$$z(x,t) \geq u(x,t) \quad \text{in} \quad B_R \times (0,T).$$

Thus

$$u(x,t) \leq \log[\frac{1}{A(R^2-r^2)^2}] < \infty \quad \text{for} \quad 0 \leq |x| \leq a < R, 0 < t < T.$$

□

Remark 4.2. From Theorem 4.1 and the upper blow-up rate estimate (3.7), it followes that, for the special case of problem (1.1) ($p = q = 1$ and λ satisfies (4.1)), the blow-up occurs only on the boundary. Therefore, we conclude that, the blow-up set of (1.1), where λ is small enough, is the same that of (1.1), where $\lambda = 0$ (see [2]).

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